A CHARACTERIZATION OF $M(G)$

BY
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Let G be a locally compact group (=locally compact Hausdorff topological group). By the measure algebra $M(G)$ of G, we mean the Banach $*$-algebra of bounded regular Borel measures on G. By the group algebra $L^1(G)$ of G, we mean the algebra of (equivalence classes of) complex-valued functions on G, summable with respect to the Haar measure on G.

Recently work has been done on the problem of characterizing those Banach algebras which are isometric and isomorphic to group algebras or measure algebras. In particular, Rieffel [6] has obtained characterizations of $M(G)$ and $L^1(G)$ for G abelian, and Greenleaf [4] has characterized $L^1(G)$ for G compact.

The main result of this paper is Theorem 1, which characterizes those Banach algebras which are isomorphic and isometric to measure algebras. For definitions and basic results regarding $M(G)$ the reader is referred to [5], and for results concerning topological vector spaces to [1].

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THEOREM 1. *Let A be a Banach algebra, S its unit ball, and S^ε the set of extreme points of S. Suppose that*

(1) *there is a Banach space E such that A is the dual of E,*

(2) *for each x in A, the mappings $y \mapsto xy$ and $y \mapsto yx$ are $\sigma(A, E)$-continuous,*

(3) *if $x \in A$ is such that $xy=0$ for all y in A, then $x=0$,*

(4) *if $x \in S^\varepsilon$, then the mapping $y \mapsto xy$ is an isometry of A onto itself,*

(5) *$S^\varepsilon \cup \{0\}$ is $\sigma(A, E)$-closed,*

(6) *there is a nonzero multiplicative linear functional p on A with the following properties: if $G=\{x \in S^\varepsilon : p(x)=1\}$ then*

(i) *for each f in E there is a g in E such that $x(g)=x(f)^-$ (the complex conjugate of $x(f)$) for all x in G,*

(ii) *for $f, g \in E$, there is an $h \in E$ such that $x(h)=x(f)x(g)$ for all $x \in G$.*

Then G is a locally compact group in the relative $\sigma(A, E)$-topology and A is isomorphic and isometric to $M(G)$. If S^ε is $\sigma(A, E)$-closed then G is compact. G is unique to within topological isomorphism. Conversely if G is a locally compact group then $M(G)$ satisfies (1)–(6).

Proof. We begin by showing that the measure algebra of a locally compact group satisfies (1)–(6). It is well known that $M(G)$ is the dual of $C_0(G)$ (the Banach

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space of continuous complex-valued functions which "vanish at infinity"). Thus (1) is satisfied. By Proposition 1 of [7], (2) is satisfied. $M(G)$ has a unit; consequently (3) holds.

For x in G let ε_x be the Dirac measure at x, and let G^ε be the set of all Dirac measures in $M(G)$. G^ε_σ is homeomorphic to G where σ is the relative $\sigma(M(G), C_0(G))$-topology. Let T be the complex numbers of absolute value 1. By Proposition 3 of [7], $TG^\varepsilon = S^\varepsilon$, and an easy calculation shows that (4) is satisfied. If G is compact, then, since G^ε_σ is homeomorphic to G and T is compact, it follows that S^ε is $\sigma(M(G), C_0(G))$-compact and hence $\sigma(M(G), C_0(G))$-closed. If G is not compact, let G^∞ be the one point compactification of G. If $(x_j : j \in J)$ is a net in G^∞ which converges to ∞, then $f(x_j) \to 0$ for each f in $C_0(G)$ so that $\varepsilon_{x_j} \xrightarrow{\sigma} 0$. Thus the mapping $x \mapsto \varepsilon_x$ has a continuous extension to G^∞ and this extension is one-one and therefore a homeomorphism of G^∞ onto $G^\varepsilon \cup \{0\}$. Thus $TG^\varepsilon \cup \{0\} = S^\varepsilon \cup \{0\}$ is $\sigma(M(G), C_0(G))$-compact and hence $\sigma(M(G), C_0(G))$-closed. Thus (5) is satisfied.

We now show that (6) is satisfied. For this let p be the linear functional on $M(G)$ defined by $p(\mu) = \mu(G)$. It follows easily that p is multiplicative. Note that $G^\varepsilon = \{\mu \in S^\varepsilon : p(\mu) = 1\}$. Thus, choosing f^- (the complex conjugate of f) for g, (i) is satisfied. (ii) is satisfied since $C_0(G)$ is an algebra.

We now prove the direct statements of the theorem. We shall divide the proof into a number of assertions.

I. A has a unit u; $u \in S^\varepsilon$ and S^ε is a group.

Since A is the dual of a Banach space, S^ε is not empty. Let $x \in S^\varepsilon$. Then by (4) the mapping T_x defined by $T_x y = xy$ is an isometry of A onto itself, hence T_x^{-1} exists and is an isometry. Put $u = T_x^{-1}x$; it follows that u is a left unit, and it is easily seen using (3) that u is a unit and $\|u\| = 1$. Kakutani has shown that the unit of a Banach algebra is necessarily in S^ε. To show that S^ε is a group it suffices to show that if $x \in A$, and the mapping $T_x : y \mapsto xy$ is an isometry of A onto itself, then $x \in S^\varepsilon$. For this, suppose that $y, z \in S$ and $x = \alpha y + (1-\alpha)z$, $0 < \alpha < 1$. Then $u = \alpha T_x^{-1} y + (1-\alpha) T_x^{-1} z$. Since T_x is an isometry, T_x^{-1} is also an isometry, consequently $T_x^{-1} y$, $T_x^{-1} z \in S$, and since $u \in S^\varepsilon$, we therefore have $y = z = x$.

II. For any $x \in S^\varepsilon$, $p(x)^- = p(x^{-1})$ and $|p(x)| = 1$.

Since p is a multiplicative linear functional on a Banach algebra with a unit u, we have $p(u) = 1$ and $\|p\| = 1$. Let $x \in S^\varepsilon$. Then $|p(x)| \leqq 1$, and, since $x^{-1} \in S^\varepsilon$, $|p(x^{-1})| \leqq 1$. If $|p(x)| < 1$,then $p(u) = p(xx^{-1}) = |p(x)|\ |p(x^{-1})| < 1$, an absurdity, so that $|p(x)| = 1$. Then $p(x)p(x)^- = 1 = p(xx^{-1}) = p(x)p(x^{-1})$ and therefore $p(x)^- = p(x^{-1})$.

We now topologize G with the relative $\sigma(A, E)$-topology and, with respect to this topology, show that G is a locally compact group.

III. G is a locally compact group and, if S^ε is $\sigma(A, E)$-closed, G is compact.

Let T be the complex numbers of absolute value 1, and consider the mapping g of $T \times G$ into S^ε defined by $g(a, x) = ax$. It is easily verified that this mapping is one-one and onto. Let S^ε_σ be S^ε taken with the relative $\sigma(A, E)$-topology. Since g is the restriction to $T\times G$ of the mapping $(\alpha, x)\to\alpha x$ of $C\times A_\sigma\to A_\sigma$, it follows that g is continuous. It is easily verified that g is open, and consequently a homeomorphism of $T\times G$ onto S^ε_σ. By (5) $S^\varepsilon\cup\{0\}$ is $\sigma(A, E)$-closed, consequently, since S is $\sigma(A, E)$-compact, $S^\varepsilon\cup\{0\}$ is $\sigma(A, E)$-compact. Therefore S^ε_σ is locally compact. Since $T\times G$ is homeomorphic to S^ε_σ, G is locally compact. Clearly if S^ε is $\sigma(A, E)$-closed then G is compact. By (2) multiplication is $\sigma(A, E)$-continuous in each variable separately so, by a theorem of Ellis [3], G is a locally compact group.

For $f\in E$, let $\hat{f}$ be the function on G defined by $\hat{f}(x)=x(f)$.

IV. $f\mapsto\hat{f}$ is a norm decreasing linear mapping of E into $C_0(G)$.

It is clear that this mapping is linear and, since $\|x\|\leqq 1$ for x in G, we have $|\hat{f}(x)|\leqq\|f\|$ so that $\|\hat{f}\|\leqq\|f\|$. Note that $\hat{f}$ is continuous. To show that $\hat{f}$ is in $C_0(G)$, first note that if S^ε_σ is compact then G is compact so that $C_0(G)=C(G)$. If G is not compact then S^ε is not compact, so that 0 is a $\sigma(A, E)$-adherence point of S^ε (since $S^\varepsilon\cup\{0\}$ is $\sigma(A, E)$-compact). Now let $\varepsilon>0$ be given and suppose $f\neq 0$. Clearly $U=\{x\in A : |x(f)|<\varepsilon\}\cap(S^\varepsilon\cup\{0\})$ is an open $\sigma(A, E)$-neighborhood of 0 in $S^\varepsilon\cup\{0\}$ so that $W=S^\varepsilon\backslash U$ is compact in S^ε. Since $T\times G$ is homeomorphic to S^ε, $g^{-1}(W)$ is compact in $T\times G$. Let K be the image of $g^{-1}(W)$ by the projection mapping $T\times G\to G$, then K is compact in G. Thus for $f\in E$, we have found a compact set K such that $|\hat{f}(x)|<\varepsilon$ for $x\notin K$ because $G\backslash K\subseteq U$.

V. Let $\hat{E}$ be the image of E in $C_0(G)$ under the mapping $f\mapsto\hat{f}$. $\hat{E}$ is dense in $C_0(G)$.

If for $x, y\in G, \hat{f}(x)=\hat{f}(y)$ for all $\hat{f}\in\hat{E}$, then $x(f)=y(f)$ for all $f\in E$; hence $x=y$, so that $\hat{E}$ separates the points of G. If $x\in G$, then $x\neq 0$, so there is an $f\in E$ such that $x(f)\neq 0$, i.e., $\hat{f}(x)\neq 0$. Thus, given $x\in G$, we can find an $\hat{f}\in\hat{E}$ such that $\hat{f}(x)\neq 0$. Further, for any $f\in E$, by (6)(i), there is a $g\in E$ such that $x(g)=x(f)^-$; i.e., $\hat{g}(x)=\hat{f}(x)^-$ for all $x\in G$. By (6)(ii), $\hat{E}$ is a subalgebra of $C_0(G)$. Thus $\hat{E}$ is a subalgebra of $C_0(G)$ which separates the points of G, does not vanish at any point of G, and is closed under complex conjugation; hence the Stone-Weierstrass theorem applies, and we may conclude that $\hat{E}$ is dense in $C_0(G)$.

Let θ be the adjoint of the mapping of $f\mapsto\hat{f}$, i.e., $\theta\mu(f)=\mu(\hat{f})$ for $\mu\in C_0(G)'=M(G)$ and $f\in E$. Note that $\theta\varepsilon_x=x$, so that by the linearity of θ we have

$$(*)\qquad \theta\left(\sum_1^n a_i\varepsilon_{x_i}\right)=\sum_1^n a_i x_i.$$

VI. θ is a norm decreasing one-one linear mapping of $M(G)$ into A, and θ is continuous as a mapping of $M(G)_\sigma$ into A_σ.

This follows from IV, V, and the general properties of adjoint mappings.

Let S^M be the unit ball of $M(G)$.

VII. $(S^M)_\sigma$ is homeomorphic to S_σ, and θ is an isometry of $M(G)$ onto A.

Since $(S^M)_\sigma$ is compact, and θ is one-one and continuous, to prove the first assertion, it suffices to show that $\theta(S^M)=S$. By VI, $\theta(S^M)\subseteq S$ so it suffices to show that $\theta(S^M)\supseteq S$. For this let $x\in S$, then since S is convex and $\sigma(A, E)$-compact, the

Krein-Milman theorem [1, Chapitre 2] applies and there is a net $(x_j : j \in J)$ such that $x_j \xrightarrow{\sigma} x$ and $x_j = \sum_{i=1}^{n_j} a_{i,j} x_{i,j}$, where $\sum_{i=1}^{n_j} a_{i,j} = 1$, $a_{i,j} > 0$ and the $x_{i,j}$ are extreme points of the unit ball in A. Putting $y_{i,j} = x_{i,j}/p(x_{i,j})$, we have $y_{i,j} \in G$ for any i, j. Considering $\mu_j = \sum_{i=1}^{n_j} a_{i,j} p(x_{i,j}) \varepsilon_{y_{i,j}}$ we see that μ_j is an element of the unit ball S^M of $M(G)$ for each j and $T\mu_j = x_j$ by (*). Since S^M is $\sigma(M(G), C_0(G))$-compact, there is a $\mu \in S^M$ and there is a subnet $(\mu_{j(i)}) \subseteq (\mu_j)$ such that $\mu_{j(i)} \xrightarrow{\sigma} \mu$. Since θ is continuous as a map of $M(G)_\sigma$ into A_σ we have that $\theta\mu_{j(i)} \xrightarrow{\sigma} \theta\mu$. $(x_{j(i)}) \subseteq (x_j)$, and $\theta\mu_{j(i)} = x_{j(i)} \xrightarrow{\sigma} x$, it follows that $\theta\mu = x$. This shows θ maps S^M onto S and hence is a homeomorphism because S^M and S are compact. Hence θ maps $M(G)$ onto A.

To show that θ is an isometry suppose there is a $\mu \in M(G)$ such that $\|\theta\mu\| < \|\mu\|$. Since S^M is mapped onto S and θ is one-to-one, $\|\theta^{-1}\| \leqq 1$. Thus $\|\mu\| = \|\theta^{-1}\theta\mu\| \leqq \|\theta\mu\| < \|\mu\|$, a contradiction.

Finally in order to show that A is isometric and isomorphic to $M(G)$, we have to show the following:

VIII. For $\mu, \lambda \in M(G)$, $\theta(\mu * \lambda) = \theta\mu\theta\lambda$.

First let $\mu, \lambda \in V$ (the linear span of the Dirac measures). Then $\mu = \sum_{i=1}^{n} a_i \varepsilon_{x_i}$ and $\lambda = \sum_{i=1}^{n} b_i \varepsilon_{y_i}$, where a_i, b_i are complex numbers. We have

$$\mu * \lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \varepsilon_{x_i} * \varepsilon_{y_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \varepsilon_{x_i y_j}$$

so that by (*);

$$(\substack{*\\*}) \qquad \theta(\mu * \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j x_i y_j = \left(\sum a_i x_i\right)\left(\sum b_i y_i\right) = (\theta\mu)(\theta\lambda).$$

Now let $\mu, \lambda \in M(G)$, then by Proposition 1 of [7] there are nets $(\mu_j : j \in J)$, $(\lambda_k : k \in K)$ in V such that $\mu_j \xrightarrow{\sigma} \mu$ and $\lambda_k \xrightarrow{\sigma} \lambda$. Then since multiplication in $M(G)$ is separately $\sigma(M(G), C_0(G))$-continuous, we have $\mu * \lambda = \lim_j (\lim_k \mu_j * \lambda_k)$. Since θ is continuous $M(G)_\sigma \to A_\sigma$ we have

$$\theta(\mu * \lambda) = \lim_j \theta\left(\lim_k \mu_j * \lambda_k\right) = \lim_j \left(\lim_k \theta(\mu_j * \lambda_k)\right)$$
$$= \lim_j \left(\lim_k \theta\mu_j \theta\lambda_k\right)$$

by the $(\substack{*\\*})$ above. By hypothesis (2) of the theorem, multiplication in A is $\sigma(A, E)$-continuous in each variable separately, thus $\lim_j (\lim_k T\mu_j T\lambda_k) = \lim_j T\mu_j T\lambda = T\mu T\lambda$ which proves the assertion.

Thus A is isometric and isomorphic to $M(G)$. The uniqueness of G is a consequence of Theorem 1 of [7].

We now examine those Banach algebras which satisfy conditions (1)–(5) of Theorem 1. We begin with some preliminary results. The following proposition is due to Greenleaf, and has appeared in [4] in a less general form.

PROPOSITION 1. *Let E be a Banach space and let N be a $\sigma(E', E)$-closed subspace of the dual E' of E. Let π be the canonical mapping $E' \to E'/N$; then π maps the unit ball of E' onto the unit ball of E'/N.*

Proof. Since N is $\sigma(E', E)$-closed, N is norm closed so that E'/N is a Banach space with the norm of an element $\pi(x)$ given by

$$\|\pi(x)\| = \inf\{\|x+n\| : n \in N\} \leqq \|x\|. \tag{1}$$

Hence $\|x\| \leqq 1$ implies $\|\pi(x)\| \leqq 1$, i.e. π maps the unit ball of E' into the unit ball of E'/N. To show that π is onto, let $\pi(x) \in E'/N$ with $\|\pi(x)\| \leqq 1$. By (1) there are $x_j \in x+N$ such that

$$\|x_j\| \leqq \|\pi(x)\| + 1/j\|x\|, \qquad j = 1, 2, \ldots.$$

The sequence $(x_j : j=1, 2, \ldots)$ is then norm bounded and therefore contained in a $\sigma(E', E)$-compact subset of E'. Thus there is a $y \in E'$ and a subsequence $(x_{j(i)}) \subseteq (x_j)$ such that $x_{j(i)} \xrightarrow{\sigma} y$. Since N and hence $x+N$ are $\sigma(E', E)$-closed, $y \in x+N$ and therefore $\pi(y)=\pi(x)$. Since the norm is $\sigma(E', E)$ lower semicontinuous, we have $\|y\| \leqq \liminf \|x_{j(i)}\| \leqq \|\pi(x)\| \leqq 1$. Thus for each element $\pi(x)$ of the unit ball of E'/N there is an element y of the unit ball of E' such that $\pi(y)=\pi(x)$.

The next proposition is also due to Greenleaf [4] in the case where G is a compact group. It should be noted that our proof is new and somewhat simpler than his.

Proposition 2. *Let G be a locally compact group; N a $\sigma(M(G), C_0(G))$-closed two-sided ideal in $M(G)$; S^ε* (resp. S^π) *the set of extreme points of the unit sphere of $M(G)$* (resp. $M(G)/N$), *and π the canonical mapping $M(G) \to M(G)/N$. Then $\pi(S^\varepsilon)=S^\pi$.*

Proof. We first show that $S^\pi \subseteq \pi(S^\varepsilon)$. Recall that $M(G)/N$ can be identified with the dual of N^0 the polar of N in $C_0(G)$, [1, Chapitre IV]. If G is not compact, let G^∞ be the one point compactification of G, and if G is compact put $G^\infty=G$. Consider $N^0 \subseteq C(G^\infty)$; N^0 is $\sigma(C_0(G), M(G))$-closed and hence norm closed. Let $\mu \in S^\pi$, then by [2, V.8.6], there is a complex number c, $|c|=1$ and an x in G such that $\mu(f) = c\varepsilon_x(f)$ for f in N^0, and this means that $c\varepsilon_x \in \pi^{-1}(\mu)$. By [7, Proposition 3] $c\varepsilon_x \in S^\varepsilon$, thus $S^\pi \subseteq \pi(S^\varepsilon)$.

If $\mu \in S^\varepsilon$, then the mapping $\lambda \mapsto \mu * \lambda$ is an isometry of $M(G)$ onto itself and it follows that $\pi(\mu)$ has the analogous property in $M(G)/N$ since π is norm decreasing. It follows as in the proof of I of Theorem 1 that $\pi(\mu)$ is in S^π. Thus $\pi(S^\varepsilon)=S^\pi$.

Theorem 2. *Let A be a Banach algebra which satisfies conditions* (1) *to* (5) *of Theorem* 1. *Then there is a locally compact group G, and a $\sigma(M(G), C_0(G))$-closed two-sided ideal N in $M(G)$ such that A is isometric and isomorphic to $M(G)/N$. Conversely if N is a $\sigma(M(G), C_0(G))$-closed two-sided ideal in $M(G)$, then $M(G)/N$ satisfies* (1) *to* (5).

Proof. Let S and S^ε be as in Theorem 1, and take G to be S^ε, then G is a locally compact group (see the proof of Theorem 1). For f in E, let $\hat{f}$ be the function on G given by $\hat{f}(x)=x(f)$. Then $f \mapsto \hat{f}$ is a norm decreasing linear mapping of E into $C_0(G)$ (see the proof of IV in Theorem 1). Let θ be the adjoint of $f \mapsto \hat{f}$, then we have that θ is a norm decreasing and continuous linear mapping of $M(G)_\sigma$ onto A_σ. The arguments to show that $\theta(\mu * \lambda)=\theta\mu\theta\lambda$ and $\theta(S^M)=S$ are similar (and easier) than those used in the proofs of VII and VIII of Theorem 1. Now let $N=\ker\theta$, then N is a weakly closed two-sided ideal in $M(G)$. Let π be the canonical mapping $M(G) \to M(G)/N$ and let θ' be the mapping $M(G)/N \to A$ such that $\theta=\theta' \circ \pi$. Clearly θ' is one-one and onto. We now show that θ' is an isometry. By Proposition 1, $\pi(S^M)$ is the unit sphere in $M(G)/N$ and since $\theta(S^M)=S$ we have $\theta'(\pi(S^M))=S$, i.e., θ' maps the unit sphere of $M(G)/N$ onto the unit sphere of A. Thus $\|\theta'\| \leqq 1$ and $\|\theta'^{-1}\| \leqq 1$, and this means that θ' is an isometry (see the calculation used in the proof of Theorem 1). This completes the proof of the first assertion.

Now let N be a $\sigma(M(G), C_0(G))$-closed two-sided ideal in $M(G)$. We shall show that $M(G)/N$ satisfies (1) to (5) of Theorem 1. $M(G)/N$ may be identified with the dual of N^0. Since N^0 is $\sigma(C_0(G), M(G))$-closed in $C_0(G)$, N^0 is norm closed and therefore a Banach space. Thus (1) is satisfied. To show that (2) is satisfied note that since N^0 is $\sigma(C_0(G), M(G))$-closed and since $N^{00}=N$, the $\sigma(M(G)/N, N^0)$-topology equals the quotient weak topology on $M(G)/N$. Thus it suffices to show that for $\lambda \in M(G)/N$, the mapping $\dot{\mu} \mapsto \lambda\dot{\mu}$ is continuous in the quotient weak topology and this is true because the quotient of a topological algebra is a topological algebra. Let S^ε (resp. S^π) be the set of extreme points of the unit ball of $M(G)$ (resp. $M(G)/N$). Let $\dot{\mu} \in S^\pi$, then by Proposition 2 there is a $\mu \in S^\varepsilon$ such that $\pi(\mu)=\dot{\mu}$. It follows that (3) and (4) are satisfied. To show that $S^\pi \cup \{0\}$ is $\sigma(M(G)/N, N^0)$-closed, note that π is weakly continuous, hence since $S^\varepsilon \cup \{0\}$ is weakly compact, and since $\pi(S^\varepsilon)=S^\pi$; $\pi(S^\varepsilon \cup \{0\})=S^\pi \cup \{0\}$ is $\sigma(M(G)/N, N^0)$-compact and hence $\sigma(M(G)/N, N^0)$-closed, so that (5) is satisfied.

Remark. We now give an example of a Banach algebra satisfying conditions (1)–(5) of Theorem 1 but which does not satisfy (6). Let T be the circle group and let

$$N = H^1(T) = \left\{\mu \in M(T) : \int e^{-in\theta}\, d\mu(\theta) = 0 \quad \text{for all } n > 0\right\}.$$

Then N is a $\sigma(M(T), C(T))$-closed ideal, so that by Theorem 2, $M(T)/N$ satisfies (1)–(5). Since $M(G)/N$ is also semisimple and commutative, there are multiplicative linear functionals on $M(T)/N$. Let π be the canonical mapping $M(T) \to M(T)/N$. If there were a locally compact group G and an isometric isomorphism $\theta: M(T)/N \to M(G)$ then $\theta \circ \pi$ would be a norm decreasing homomorphism of $M(T)$ onto $M(G)$. It is shown in [7; §4] that $\theta \circ \pi(L^1(T))=L^1(G)$. In [4; §2] Greenleaf has shown that $\pi(L^1(T))$ is not isometrically isomorphic to the group algebra of any locally compact group. Consequently $M(T)/N$ cannot be the measure algebra of a locally compact group.

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